Pricing and risk of financial products

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- AAA bonds are typically regarded as risk-free investment.
- Only examples: Government bonds of Australia, Canada, Denmark, Germany, Luxembourg, the Netherlands, Norway, Singapur, Sweden and Switzerland

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- Latvian government bonds are rated A-.
- Bonds and equity of any company are always regarded as risky asset.

A simple market model

One-step binomial tree

Let us assume that the price of a stock at $t = 0$ is equal to $S(0) = 100€$. The price at $t = 1$ is, of course, not known to the investor. There are two possibilities which can happen, namely

$$S(1) = \begin{cases} 140€ & \text{with probability } 50\% \\ 80€ & \text{with probability } 50\%. \end{cases}$$

Thus, the return of the stock is given by

$$K(1) = \frac{S(1) - S(0)}{S(0)} = \begin{cases} +40\% & \text{with probability } 50\% \\ -20\% & \text{with probability } 50\%. \end{cases}$$

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Risk measures (1/4)

Exercise

Let the interest rate be $r = 2\%$ and let the risky return $K(1)$ be

$$K(1) = \begin{cases} 4\% & \text{with probability } 40\% \\ -2\% & \text{with probability } 60\%. \end{cases}$$

How do we measure the inherent risk of the stock?

Risk measures (2/4)

Variance and standard deviation

The variability of a random variable X with mean μ is typically measured by the **variance** $\sigma^2(X) := E((X-\mu)^2)$ or its square root, the **standard deviation**, $\sigma = \sqrt{\sigma^2(X)}$.

Example

The expected return of the last example is

$$E(K(1)) = 0.04 \cdot 0.4 + (-0.02) \cdot 0.6 = 0.004$$

and the variance is

$$\sigma^2(K(1)) = (0.04 - 0.004)^2 \cdot 0.4 + (-0.02 - 0.004)^2 \cdot 0.6 = 0.000864.$$

Finally, we calculate the standard deviation as $\sigma \approx 0.029$.

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Risk measures (3/4)

Risk measure

Let $\mathcal{L}$ be the set of random variables. A **risk measure** is a function $\rho : \mathcal{L} \to \mathbb{R} \cup \{\pm\infty\}$.

Desirable properties

Let X, Y be two arbitrary random variables, $a \in \mathbb{R}$ and $\lambda \geq 0$. A risk measure ρ is called **coherent** if it satisfies the following properties

(i) **Translation invariance:** $\rho(X + a) = \rho(X) + a$

(ii) **Positive homogenity:** $\rho(\lambda X) = \lambda \rho(X)$.

(iii) **Monotonicity:** If $P(X \leq Y) = 1$, then $\rho(X) \leq \rho(Y)$.

(iv) **Subadditivity:** $\rho(X + Y) \leq \rho(X) + \rho(Y)$

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Risk measures (4/4)

Example

The variance satisfies none of the properties (i) - (iv). For instance, we know that for $a, b \in \mathbb{R}$ we have

$$\sigma^2(aX + b) = a^2\sigma^2(X)$$

showing that the variance is neither translation invariant nor positive homogenous. The standard deviation only satisfies (ii).

Value-at-Risk (1/3)

Value-at-Risk

Let $\alpha \in (0,1)$ be a fixed number and let X be a random variable. The cumulative distribution function of X is denoted by $F_X(\cdot)$. The **value-at-risk** is the α-quantile of X, i.e.

$$\mathrm{VaR}_\alpha := \inf\{x \in \mathbb{R} \mid F_X(x) \geq \alpha\}.$$

Value-at-Risk (2/3)

Example

A risky stock can realize the following gains or losses within one period

Loss	−500€	−300€	0€	100€	200€	400€	900€
Probability	15%	10%	35%	20%	15%	2%	3%

Calculate the value-at-risks for $\alpha = 0.9, 0.95, 0.96, 0.99$.

Value-at-Risk (3/3)

Theorem

The value-at-risk is translation-invariant, positively homogeneous and monotone.

Example

Consider two different stocks X_1 and X_2 of two companies.

Scenario	A	B	C
Loss X_1	100€	0€	−1€
Loss X_2	0€	100€	−1€
Probability	0.006	0.006	0.988

What is the value-at-risk for $\alpha = 0.99$ of X_1, X_2 and $X_1 + X_2$?

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Gambling vs. System (1/4)

Lottery

Let us consider a very simple lottery. We draw 3 out of 7 numbers. The player wins a certain amonut of money whenever he chose 2 or 3 numbers correctly. Let the payoff function be the following

Pair (deuce)	50 €
Triple	150 €

A lottery tickets costs 20 €.

Gambling vs. System (2/4)

Lottery

The expected value of a lottery ticket is

$$E[X] = P(X = 2) \cdot 50 + P(X = 3) \cdot 150 = 12/35 \cdot 50 + 1/35 \cdot 150$$
$$= 21\frac{3}{7}.$$

Therefore, the player decides to buy 7 tickets with expected value.

$$E[7X] = 150.$$

Question

How would you fill out the 7 lottery tickets?

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Gambling vs. System (3/4)

1. Tactics (Gambling)

The probabilty to have a triple is $\frac{7}{35} = 20\%$. The probability to have deuces is binomial distributed with $n = 7$ and $p = \frac{12}{35}$. That means:

0 deuces	5,3 %
1 deuce	19,3 %
2 deuces	30,3 %
3 deuces	26,3 %
4 deuces	13,7 %
5 deuces	4,3 %
6 deuces	0,7 %
7 deuces	0,1 %

Gambling vs. System (4/4)

2. Tactics (Quasi-Monte Carlo Approach)

We select the following 7 lottery tickets

$$124, 135, 167, 257, 347, 236, 456.$$

The probabilities of deuces are the following:

0 deuces	20 %
3 deuces	80 %

It is easy to check that we have a triple if and only if we have 0 deuces. We thus always get a payoff of 150€. We eliminated the risk.

Mathematical description

Find an approximation of

$$E[X] = \int x f(x) \, \mathrm{d}x,$$

which converges as quickly as possible.

The central question of Quasi-Monte Carlo

Find a sequence of points $(x_i)_{i \in \mathbb{N}}$, such that the integral

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) \to \int f(x) \, \mathrm{d}x$$

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