Pricing of Financial Products

Christian Weiß

February 28, 2018

These are the lecture notes of a course held at Riga Technical University in February 2018. The author acknowledges their kind hospitality. Moreover he wishes to thank Martin Scholtes for his useful comments on an earlier version of these notes.

Table of contents

1 Modelling financial markets

Would you prefer to receive 100€ today plus 900€ in ten years or 200€ next year plus 800€ in nine years? Financial decisions like this have to be taken all the time by banks, insurance companies, the treasury secretary and every single one of us. Due to their importance, financial decisions should, of course, be based on rational reasoning. Therefore, **financial mathematics** mainly treats the question how to find a price for financial products respectively cash flows.

The roots of financial mathematics go back to the dissertation of Louis Bachelier (1870–1946). Therein, he introduced the present value of future payments. The idea behind it can be explained very easily: most people would prefer receiving 100€ today to receiving the same amount next year. The main reason is that money received in the future cannot be spent today (just think about the cool things you would do with 100€ *today* and you will probably agree). In this chapter we will give a precise definition of the present value of a future risk-free cash flow. By risk-free we mean that we will receive (pay) the cash flow without any doubt, or to formulate in mathmatical terms with probability 1. In order to make sure that the pricing of different assets is consistent (Theorem 1.6) we have to furthermore spend some thought on the way we model markets.

1.1 Basic pricing principles

Suppose you receive a fixed amount of money today, say $P = 100€$. If you do not have any particular plans with the money, you can put it on your bank account. There, it will earn some **interest** $r > 0$. Let us denote the amount of money on your bank account at time t by $V(t)$.[1] Using the notation we just introduced, this means your bank account will today have the value

$$V(0) = P = 100$$

and next year

$$V(1) = 100 + r \cdot 100 = (1 + r) \cdot 100.$$

If you leave the money on the bank account for another year, it will rise to

$$V(2) = (1 + r) \cdot 100 + r \cdot (1 + r) \cdot 100 = (1 + r)^2 \cdot 100$$

[1] Throughout these notes the unit of time will be one year.

or, in general, at any point t in time to

$$V(t) = (1+r)^t \cdot 100. \tag{1.1}$$

Example 1.1: Interest rate

The interest rate r is equal to 2%. After three years an amount of $1'380$€ is needed. How much money do we have to put on the bank account at $t = 0$?

If we initially put an amount of P on the bank account, it will rise to

$$V(3) = 1.02^3 \cdot P$$

after three years. Thus, we need to solve the equation

$$1'380 = 1.02^3 \cdot P \qquad \Leftrightarrow \qquad P = \frac{1'380}{1.02^3} \approx 1'300.40€$$

implying the bank account has to be fed with $1'300.40$€ at $t = 0$.

If we look at Example 1.1 once again, we discover that we have just derived a very important principle: we now know which price to pay for a payment of $1'380$€ in 3 years. In other words, the **present value** of $1'380$€ in three years is $1'300$€. To formulate this insight as a mathematical formula we write

$$V(0) = V(t) \cdot (1+r)^{-t}. \tag{1.2}$$

The factor $(1+r)^{-t}$ is called **discount factor**. Furthermore, we can calculate the **rate of return** of our investment which is defined by

$$K(t) := \frac{V(t) - V(0)}{V(0)}.$$

In our example, this means

$$K(t) = \frac{(1+r)^t \cdot 100 - 100}{100} = (1+r)^t - 1$$

and in particular $K(1) = r$.

Now we come to a more general setting: Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of amounts of money. If K_n has positive sign, it corresponds to money we earn. In contrast, negative sign corresponds to money we have to pay to another party.

Example 1.2: Present value

The payment structure of a specific **bond** *is the following. Its nominal value is* 500€ *and its* **maturity** *is* 5 *years. In addition, a coupon of* 4% *is paid out every year. The market interest rate is* 2%. *Which price should we be willing to pay for the bond?*

Recall that the nominal value of a bond is paid out at maturity. This means, we have the following positive payments

$$K_1 = 4\% \cdot 500 = 20, \quad K_2 = 20, \quad K_3 = 20, \quad K_4 = 20, \quad K_5 = 500 + 20 = 520.$$

The present value of K_1 *is* $\frac{20}{1.02}$, *the present value of* K_2 *is* $\frac{20}{1.02^2}$ *and so on. Hence, the present value of the bond is*

$$\frac{20}{1.02} + \frac{20}{1.02^2} + \frac{20}{1.02^3} + \frac{20}{1.02^4} + \frac{520}{1.02^5} \approx 547.13€,$$

and the fair price of the bond is 547.13€.

Using the finite geometric series, it can be easily deduced from Example 1.2 that, if a payment C is made once a year for n years, then its present value is

$$\frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n} = C\frac{1-(1+r)^{-n}}{r}. \tag{1.3}$$

Most importantly, we see that the present value of a bond increases if the interest rate goes down. Although the market model is still rather simple, this fact perfectly explains why the prices of all assets (equity, bonds, property,...) massively rise whenever the European Central Bank (ECB) announces that they will lower interest rates.

Example 1.3: Investment decision

Assume that the interest rate is equal to $r = 1\%$. *Which investment would you prefer:*

(i) *a constant yearly payment from today on for the next* 9 *years of* 100€ *or*

(ii) *a lump sum of* 950€ *today?*

Obviously, the present value of the lump sum is 950€. *In order to calculate the present value of (i) we use* (1.3) *with* $C = 100$, $r = 0.01$ *and* $n = 9$ *implying a present value of* 856.60€. *Furthermore we have to add the* 100€ *we get already today such that the present value of (i) is* 956.60€. *From a financial mathematical point of view it would thus be rational to choose (i).*

Exercise 1.1

The market interest rate is $r = -1\%$. Calculate the market value of a zero-coupon bond with maturity 10 years and nominal value 170€.

Exercise 1.2

Suppose the interest rate is equal to r and consider a bond with maturity n years. Its nominal value is denoted by F and the coupon is i (in percent of the nominal, i.e. $C = iF$). Calculate the present of value of the bond!

1.2 Market assumptions

So far we have exclusively considered a single (risk-free) asset and calculated its present value. This is not a realistic setting since there are plenty of different assets available on real-life market. So let us turn to the following market model: Suppose that two assets are traded. One of them is a (risk-free) bond and the other is a (potentially risky) stock. The value of the bond at time t is denoted by $A(t)$. Recall that we have just discussed in Section 1.1 how to calculate $A(t)$ for any t.[2] The value of the stock is denoted by $S(t)$. Economically only values $A(t) \geq 0$ $(t) \geq 0$ for all t make sense. We do not have any formula at hand to find the price of $S(t)$ and postpone a further discussion of this topic to Chapter 2. For now, we will concentrate on further assumptions we should impose on the market model.

Our portfolio $V(t)$ at time t consists of x stock shares and y bonds such that its value is

$$V(t) = xS(t) + yA(t).$$

We allow any shares $x, y \in \mathbb{R}$ such that our first market assumption is that the stock and the bonds are arbitrarily **divisible**, e.g. we may buy 17.3% of a bond. The fact that no bounds on x, y are imposed constitutes the **liquidity** of the market. Whenever he owns an assset, i.e. $x > 0$ or $y > 0$ the investor has **long position**. In contrast $x < 0$ or $y < 0$ are a **short position**. This means that the investor borrows the asset, sells it, and uses the gained money for some other investment.

Example 1.4: Short sell

The price of a bond at $t = 0$ is equal to 5€ and the price of the stock is 10€. An investor wants to buy 30 bonds today. In order to finance the price of 150€ he short sells 15 shares of the stock.

[2] To calculate $A(t)$ we only need some easily available information namely market interest rate, maturity of the bond, its nominal and coupon.

A short sell might seem to be a bit strange process: How can you be allowed to sell an asset you do not own but only borrowed it? Nevertheless, short sales are indeed legal in reality under some limitations. The main point is that the investor must always have sufficient resources to fulfill the resulting obligations of a short sale, e.g. dividend. In particular, the investor must always be able to close the short position implying that the value of his portfolio is always non-negative, i.e. $V(t) \geq 0$ for all t. In that case the portfolio is called **admissible**.

During our discussion of bond prices we have already tacitly used the **no-arbitrage condition**. It simply says that no riskless profit, called **arbitrage**, can be made without initial capital. This is the most fundamental assumption about markets and can be justified in the following way: Any arbitrage opportunity is immediately exploited by some trader on the market such that it disappears again immediately. Hence, no (theoretical) trading strategy can be based on arbitrage. To put it in mathematical terms, the no-arbitrage condition states that there does not exist a portfolio with initial value $V(0) = 0$ and $V(t) > 0$ for some t with non-zero probability.

Example 1.5: Arbitrage strategy

Let the interest rate be $r = 2\%$. We consider a zero-coupon bond with nominal $120€$ and maturity 4 years and suppose that its market price is exactly $110€$.

On this fictitious market, an example of an arbitrage strategy is then given as follows: At $t = 0$ we borrow $990€$ on the market and buy 9 bonds. So in $t = 4$ we have to pay back

$$990€ \cdot 1.02^4 \approx 1071.61€$$

for the loan. On the other hand we receive $9 \cdot 120€ = 1080€$ as the nominal payment of the bonds. In total we make a gain of $1080€ - 1071.61€ = 8.39€$ in $t = 4$.

As Example 1.5 indicates, the no-arbitrage condition fixes the prices of bonds. More generally, a very important consequence of the no-arbitrage principle is the law of one price.

Theorem 1.6: Law of one price

Let V_1 and V_2 be two portfolios with $V_1(t_0) = V_2(t_0)$ for some t_0 with probability 1. If there are no arbitrage opportunities, then $V_1(t) = V_2(t)$ holds for all $0 \leq t \leq t_0$.

Proof. We give an indirect argument and show that there exists arbitrage opportunity if the prices do not coincide for all $0 \leq t \leq t_0$. Thus, assume the claim is wrong implying there exists t with $0 \leq t \leq t_0$ such that $V_1(t) \neq V_2(t)$. Without loss of generality we may assume $V_1(t) < V_2(t)$. Then we buy a unit of $V_1(t)$ (long position) and sell a unit of $V_2(t)$ (short position) at time t. This can be done without initial capital. The difference $V_2(t) - V_1(t)$ is invested in x units of $A(t)$. At time t_0 we liquidate the portfolio. Since V_1 and V_2 have the same price in t_0, our long and short position cancel out. The positive value $xA(t_0)$ is our riskless arbitrage gain. This is a contradiction. □

Exercise 1.3

The interest rate r is 1%.
(i) Use an arbitrage argument to calculate the nominal of a zero-coupon bond with maturity 4 and market price $C \cdot 1.01^{-4}$.
(ii) Do the same exercise as in (i) for a bond with constant coupon C, nominal $F = C$, maturity 2 years and market price $C \cdot 1.01^{-2}$.

2 Stochastic Stocks

Although, it is widely accepted in financial industry to regard the gains of AAA bonds as risk-free, this is a very rare case. In fact, only the government bonds of Australia, Canada, Denmark, Germany, Luxembourg, the Netherlands, Norway, Singapur, Sweden and Switzerland are unanimously regarded as AAA according to the three big rating agencies Fitch, Moody's and Standard & Poors, see [Boe18]. For instance, the current rating of Latvian government bonds is A- at all three credit rating agencies. And not even very optimistic traders would regard bonds or equity of any company as risk-free. A good recent example why not to do that is the German car company Volkswagen which was the most valuable (according to market capitalization) company worldwide a few years ago but got into huge trouble in 2017 due to the *dieselgate* scandal.

These observations underpin the need to also find a method for describing risky assets. In this chapter, we will therefore present two different models which are very often used to treat risky assets. Moreover, we aim to better understand what the term *risk* precisely means.

2.1 Binomial tree model

Recall that our market model consists of two assets. On the one hand we have the (risk-free government) bond $A(t)$ and on the other hand the (risky) stock $S(t)$. In this section, we restrict to discrete time. Thus, we only consider the values $A(0), A(1), A(2), \ldots$ and $S(0), S(1), S(2), \ldots$. In practice, this would mean that we only look at year-end value of the assets.[1] To make the whole story even more accessible let us at first look at only two time periods, i.e. $t \in \{0, 1\}$.

Example 2.1: One-step binomial tree

Let us assume that the price of the stock at $t = 0$ *is equal to* $S(0) = 100$€. *The price at* $t = 1$ *is, of course, not known to the investor. In mathematical language we say that* $S(1)$ *is a random variable. Furthermore, let us assume that there*

[1] This might even be a concept which makes more sense than following stock prices on a daily basis or even every second.

are two possibilities which can happen, namely

$$S(1) = \begin{cases} 140€ & \text{with probability } 50\% \\ 80€ & \text{with probability } 50\%. \end{cases}$$

The situation can be visualized by Figure 2.1 which is called a **binomial tree**

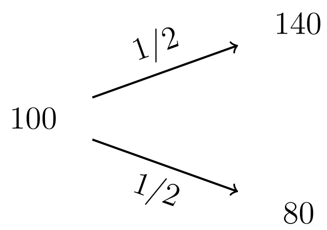

Figure 2.1 One-step binomial tree

The return of the stock is given by

$$K(1) = \frac{S(1) - S(0)}{S(0)} = \begin{cases} +40\% & \text{with probability } 50\% \\ -20\% & \text{with probability } 50\%. \end{cases}$$

Although Example 2.1 describes a not so complicated situation, it is not immediately clear how to decide if it is worth to buy a share of the stock or not. A well-known solution for the task is to compare the expected price with the risk-free investment. The idea behind this approach is the following: If an investor with lots of money buys many different (uncorrelated) stocks of the same type, he will gain 80€ in approximately 50% of the cases and 140€ else (law of large numbers). In the excel template *Example_2_1.xlsx*, we simulate 1000 different stocks all having the cash flow profile from Example 2.1. By using random numbes (Column B) and calculating the corresponding value of $S(1)$ (Column C) we see that the medium return of the simulation is approximately equal to the expected value

$$E(S(1)) = 140€ \cdot 0.5 + 80€ \cdot 0.5 = 110€.$$

The alternative investment to buying a share of the stock would have been to invest 100€ into the risk-free bond. If $r > 10\%$, then the risk-free bond would earn more money than the expected value of the stock resulting in the investment decision

$$\begin{cases} \text{buy bond} & \text{if } r > 10\% \\ \text{buy stock} & \text{if } r < 10\%. \end{cases}$$

On the one hand this investment decision seems to be rationally justified on the other hand you might have the feeling that something is wrong with it. We will leave this thought aside for now and postpone a detailed discussion to Section 2.2.

Example 2.2: General one-step binomial tree

Let us give a more general description of the situation in Example 2.1. Instead of describing the possible stock prices $S(0), S(1)$ *we turn to the return* $K(1)$*. The stock may either go up (u) or down (d). The probability for a rise in value is* $p \in [0,1]$ *such that we end up with*

$$K(1) = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p \end{cases}$$

or as a binomial-tree for a unit stock, i.e. $S(0) = 1$

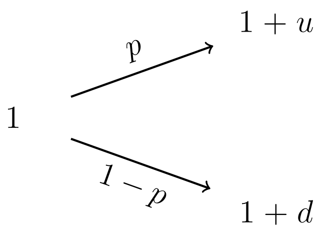

Figure 2.2 General one-step binomial tree

As usual, we denote the interest rate by r*. Is there anything we can say about d and u just having this information? Thinking back to what we have learned in Section 1.2, there is an interesting consequence of the no-arbitrage condition: The binomial tree model admits no arbitrage if and only if* $d < r < u$*. For a proof, we refer the reader to [CZ03], Proposition 4.2.*

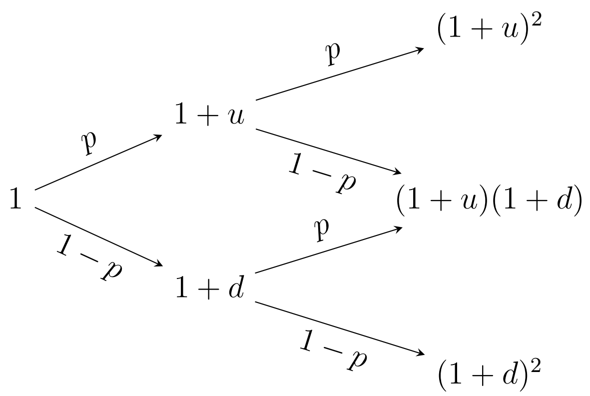

Figure 2.3 Two-step binomial tree

However, we do not live in a world where time just goes on only for one period (corresponding to one year). Therefore, the binomial model should be made ready for several years. This is a straight-forward task. We can append another binomial tree to the vertices on the right. Inductively, we get a binomial tree for n time steps $S(1), S(2), \ldots, S(n)$. For instance, a two-step version of the binomial tree in Example 2.2 looks like in Figure 2.3. The stock can go up or down in the first time step and the same can happen in the second time step. For instance, the probability of having a stock value of $(1+u)(1+d)$ at $t = 2$ is $p(1-p)+(1-p)p = 2p(1-p)$. The two-step version is implemented in the excel template *Example_Two_Step_Binomial.xlsx.*

Exercise 2.1

Consider the two-step tree given by Figure 2.4

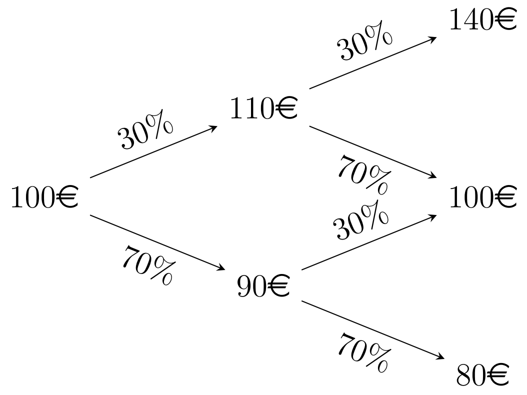

Figure 2.4 Two-step binomial tree

Calculate the expected value of the stock at $t = 1$ and $t = 2$.

Exercise 2.2

A two-step binomial tree as in Figure 2.3 has the possible values 32€, 28€ and $x < 28$€ at $t = 2$. Can you complete the tree? Can this be done uniquely?

Exercise 2.3

Consider a four-step binomial tree, where the stock goes up with probability p and goes down with probability $1 - p$. What is the probability that the value of the stock at $t = 4$ is equal to $S(4) = (1 + u)^3(1 + d)$? Which well-known probability distribution explains your calculation?

2.2 Let's talk about ... risk

Hang on a second! Let us take a step back and think again about what we just said: Which asset would you personally buy if you can choose between a risk-free return of $r = 9.9\%$ and a risky asset with a return of either 5% or 15% with equal probability? Either you can get 109.90€ for sure or buy an asset with an expected value of 110€. Psychological experiments found out that most people would prefer the risk-free gain, see e.g. [Kah12]. In other words, a typical (risk-averse) investor would require a compensation for risk, i.e. the expected gain of the stock has to be greater than the risk-free one.

It might sound a bit weird at first sight but it is possible to describe mathematically what it means if a rational[2] investor finds a risk-free return of 10% equally good as a risky one with expected value of 10%.

Example 2.3: Risk-neutral measure

We return to Example 2.2, where we discussed the general one-step binomial tree model with return

$$K(1) = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p \end{cases}$$

and risk-free interest r. A rational investor who is indifferent between the two investment options assesses the risky asset with his own probability p_* instead of p. In order to satisfy the indifference property, the variable p_* has to fulfill the equation

$$E_*(K(1)) = p_* u + (1-p_*)d = r.$$

This is equivalent to

$$p_* = \frac{r-d}{u-d}.$$

In this case p_* is called **risk-neutral probability** *and E_* is called* **risk-neutral expectation**. *If $p = p_*$ then the investor is risk-neutral.*

Exercise 2.4

Let the interest rate be $r = 2\%$ and let the risky return $K(1)$ be

$$K(1) = \begin{cases} 4\% & \text{with probability } 20\% \\ -0.5\% & \text{with probability } 80\%. \end{cases}$$

[2] We use the word *rational* here and in the following as a synonym for a profit-maximizing person. It is seriously doubted if this is the correct point of view both from a psychological perspective (see e.g. [Fin06], [Kah12]) as well as from an economical (see e.g. [Sen77]).

Calculate the risk-neutral probability!

Exercise 2.5

Show that $d < r < u$ if and only if $0 < p_* < 1$.

We have already made use of the term *risk* many times in these notes without stating what we understand by it. It is widely discussed how to give a precise definition. Here come three examples:

> A probability or threat of damage, injury, liability, loss, or any other negative occurrence that is caused by external or internal vulnerabilities, and that may be avoided through preemptive action (from [Bus18]).
>
> Risk implies future uncertainty about deviation from expected earnings or expected outcome (from [Eco18]).
>
> The Concise Oxford English Dictionary defines risk as 'hazard, a chance of bad consequences, loss or exposure to mischance' (from [MFE05]).

All these quotes give us a glimpse of the truth. Probably it is impossible to give a one-sentence definition of *risk* but there are two points our examples (and also most other definitions) have in common, namely *uncertainty* and the potential of *loss*.

We now turn to the question how to capture risk in mathematical language. The variability (corresponding to the uncertainty) of a random variable X with mean μ is typically measured by the **variance**

$$\sigma^2(X) := E((X-\mu)^2)$$

or its square root, the **standard deviation**

$$\sigma = \sqrt{\sigma^2(X)}.$$

The advantage of the standard deviation is that its unit is the same as the one of X while units are squared in the variance.

Example 2.4: Variance and Standard deviation

Let the risky return $K(1)$ be

$$K(1) = \begin{cases} 4\% & \text{with probability } 40\% \\ -2\% & \text{with probability } 60\%. \end{cases}.$$

The expected return is

$$E(K(1)) = 0.04 \cdot 0.4 + (-0.02) \cdot 0.6 = 0.004$$

and the variance is

$$\sigma^2(K(1)) = (0.04 - 0.004)^2 \cdot 0.4 + (-0.02 - 0.004)^2 \cdot 0.6 = 0.000864.$$

Finally, we calculate the standard deviation as

$$\sigma = \sqrt{\sigma^2(K(1))} \approx 0.02939.$$

The classical theory for portfolio management is based on measuring variance or standard deviation, see e.g. [Alb07], [CZ03], [Kal17]. This idea goes back to the dissertation of Nobel laureate Markowitz [Mar52] who was himself not really satisfied with his approach. If we reconsider what we just learned about risk we can understand why: Typcially risk is about potential *losses* and not about potential *gains*. However, the variance measures deviations to the down as well as deviations to the up.

Hence, let us face risk from a different perspective, namely an axiomatic one. At first, it makes sense to think about what a risk measure does and take this as an abstract definition.[3]

Defintion 2.5

Let $\mathcal{L}$ be the set of random variables. A **risk measure** is a function $\varrho : \mathcal{L} \to \mathbb{R} \cup \{\pm\infty\}$.

Example 2.6: Risk measures

Variance and standard deviations are risk measures. Another risk measure is the function which assigns the value $+1$ *to every random variable.*

So far a risk measure is simply a function that assigns a value to every random variable. Without any further restrictions, this is not very useful for measuring risk because we may choose an arbitrary function. Therefore, we should ask: Which properties would be desirable when measuring risk (of one or several assets)?

[3] If we want to be really precise we have to fix a probability space $(\Omega, \mathcal{F}, P)$ at first and refer to it in the following. We leave away this technical aspect to make the presentation easier to understand for readers who are not mathematicians.

Defintion 2.7

Let X, Y be two arbitrary random variables, $a \in \mathbb{R}$ and $\lambda \geq 0$. A risk measure ϱ is called **coherent** if it satisfies the following properties

(i) **Translation invariance:** $\varrho(X + a) = \varrho(X) + a$

(ii) **Positive homogeneity:** $\varrho(\lambda X) = \lambda \varrho(X)$.

(iii) **Monotonicity:** If $P(X \leq Y) = 1$, then $\varrho(X) \leq \varrho(Y)$.

(iv) **Subadditivity:** $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$

Let us interpret the definition and argue by that that these are indeed properties of a *good* risk measure. If a risk measure is *translation invariant* and any fixed number is added to a random variable, the risk measure increases by the same amount. In other words, a secure loss of a€ increases the risk measure by a€ because risk measures are usually applied to losses (see Example 2.10) . If you multiply the random variable by a positive number, then a *positive homogeneous* risk measure is multiplied by the same number. It makes absolutely sense to restrict to positive numbers since multiplication by a negative number corresponds to switching gains and losses. If the outcome of one random process is smaller than that of another one for sure, the risk measure should reflect it (*monotonicity*). Finally, the *subadditivity* property catches the idea that two different risks hedge if they are treated together instead of looking at each risk separately.

Example 2.8: Coherent risk measure

The variance satisfies none of the properties (i) - (iv). For instance, we know that for $a, b \in \mathbb{R}$ we have

$$\sigma^2(aX + b) = a^2\sigma^2(X)$$

showing that the variance is neither translation invariant nor positive homogeneous. The standard deviation only satisfies (ii).

Exercise 2.6

Find a counterexample proving that also properties (iii) and (iv) do not hold for variance and standard deviation.

This observation shows that neither variance nor standard deviation are very smart risk measures. Moreover, it explains why neither of it found its way into the two most important current European laws regarding regulation of the financial sector, namely Solvency II for insurance companies, see [EC15], and Basel III for banks, see [EC13]. Which risk measure do these laws apply instead?

Defintion 2.9

Let $\alpha \in (0,1)$ be a fixed number and let X be a random variable. The cumulative distribution function of X is denoted by $F_X(\cdot)$. The **value-at-risk** is the α-quantile of X, i.e.

$$\mathrm{VaR}_\alpha := \inf\left\{x \in \mathbb{R} \mid F_X(x) \geq \alpha\right\}.$$

Recall that the cumulative distribution function of a random variable X is defined by $F_X(x) = P(X \leq x)$. From an economic point of view, the value-at-risk is thus the loss which will not be exceeded with probability $\alpha \cdot 100\%$. The abstract definition becomes clearer from the following example.

Example 2.10: Calculation of value-at-risk

(i) *A risky stock can realize the following gains or losses within one period*

Loss	−500€	−300€	0€	100€	200€	400€	900€
Probability	15%	10%	35%	20%	15%	2%	3%

By definition we calculate the following value-at-risks

$$VaR_{0.9} = 200€, \qquad VaR_{0.95} = 200€, \qquad VaR_{0.96} = 400€, \qquad VaR_{0.99} = 900€.$$

(ii) Let $a \in (0,1)$ be an arbitrary fixed number. Recall that a random variable X is normally distributed with expected value μ and variance σ^2 if its density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}}.$$

The cumulative distribution function is given by

$$\Phi_{\mu,\sigma}(x) = \int_{\infty}^{x} f(t)\,\mathrm{d}t.$$

For calculating the value-at-risk, we write down its definition differently

$$\begin{aligned}\alpha &= \Phi_{\mu,\sigma}\left(VaR_\alpha\right)\\ &= \Phi_{0,1}\left(\frac{VaR_\alpha - \mu}{\sigma}\right),\end{aligned}$$

where we applied the standardization technique for random variables in the last step. From this we deduce

$$VaR_\alpha = \mu + \Phi_{0,1}^{-1}(\alpha)\sigma.$$

The value of $\Phi^{-1}(\alpha)$ can be looked up in a table and is also implemented in Excel and most computer algebra systems.

Exercise 2.7

Calculate the value-at-risk for the quantile $a = 0.92$ of the following losses X.

(i) X is a discrete random variable with

Loss	−1.200€	−800€	−100€	0€	50€	100€	8.888€
Probability	12%	60%	2%	10%	4%	7%	5%

(ii) X is a random variable uniformly distributed on $[0, 1]$, that means its density is given by

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{else.} \end{cases}$$

In comparison to variance and standard deviation, the value-at-risk performs much better. It can be easily proven that it fulfills the first three of the four properties we regarded as desirable.

Theorem 2.11

The value-at-risk is translation-invariant, positively homogeneous and monotone.

Exercise 2.8

Proof that the value-at-risk is monotone.

However, the value-at-risk is in general not subadditive and hence not a coherent measure.[4]

Example 2.12: Non-coherence of value-at-risk

Consider two different stocks X_1 and X_2. There are three events which may happen: Scenario A describes a situation where the company emitting stock X_1 gets into economic trouble. This will generate big losses on their side but does not affect the other company. Scenario B describes the opposite situation which harms the company emitting X_2. Finally, in scenario C both companies prosper.

[4] Therefore, the Swiss regulatory authority FINMA decided to use a different risk measure instead of the value-at-risk for the Swiss solvency test, [Eid17]. It is called **expected shortfall** and can be calculated with similar effort as the value-at-risk.

Scenario	A	B	C
Loss X_1	100€	0€	-1€
Loss X_2	0€	100€	-1€
Probability	0.006	0.006	0.988

The values-at-risk for $\alpha = 0.99$ are thus $VaR_{0.99}(X_1) = VaR_{0.99}(X_2) = 0$€ for both companies. Now we look at the sum of the losses $X_1 + X_2$

Scenario	A	B	C
Loss $X_1 + X_2$	100€	100€	-2€
Probabilty	0.006	0.006	0.988

resulting in

$$VaR_{0.99}(X_1 + X_2) = 100\text{€} > 0\text{€} = VaR_{0.99}(X_1) + VaR_{0.99}(X_2).$$

As we see, the value-at-risk is in general indeed not subadditive.

Although this disadvantage leaps to the eye, the value-at-risk is widely accepted as can be perceived by its application in the regulation of banks and insurance companies. Why? The first part of the answer is the one that applies very often in probability theory and statistics, namely the *central limit theorem*, see e.g. [Dur91], Theorem (4.1). If a random process $X_1, X_2, \ldots$ is repeated many times then under some mathematically rather mild conditions[5] the sum $X_1 + X_2 + \ldots + X_n$ is approximately normally distributed. That is the reason why normally distributed variables occur so often in reality and why it makes sense to focus on this case in the context of regulation. The second part of the answer is that the value-at-risk is coherent for normally distributed random variables.

Theorem 2.13

For two normal random variables X, Y their sum is also normally distributed and for each $\alpha \in (0, 1)$ the inequality

$$\mathrm{VaR}_\alpha(X + Y) \leq \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$$

holds, i.e. the value-at-risk is subadditive for normal random variables.

We provide an excel template, called *VaR.xlsx*, where the value-at-risk of a random variable is calculated theoretically and simulated given any user input for μ, σ and α. Since random numbers are used for the simulation, it is common to speak of a **Monte Carlo simulation** of the value-at-risk. By the Monte Carlo simulation

[5] These conditions are that $X_1, X_2, \ldots$ are independent and identically distributed with $E[X_i] = \mu$ and $\sigma^2(X_i) \in (0, \infty)$.

it can be verified that the formula which was derived in Example 2.10 by purely theoretical arguments, can indeed be (approximately) observed in reality.[6]

2.3 Geometric Brownian motion

In Section 2.1 we have introduced a first discrete time model for the movement of a risky stock. We discussed how to calculate the expected value and defined the associated risk-neutral measure. However, we have not challenged whether the model reflects reality approriately. In order to assess its quality, it is a first idea to compare a single path of a binomial-tree simulation to a real life stock chart. In the excel template *Binomial_Tree_Model.xlsx* we can simulate the movement of a stock for 60 time steps. A typical outcome is shown in Figure 2.5.

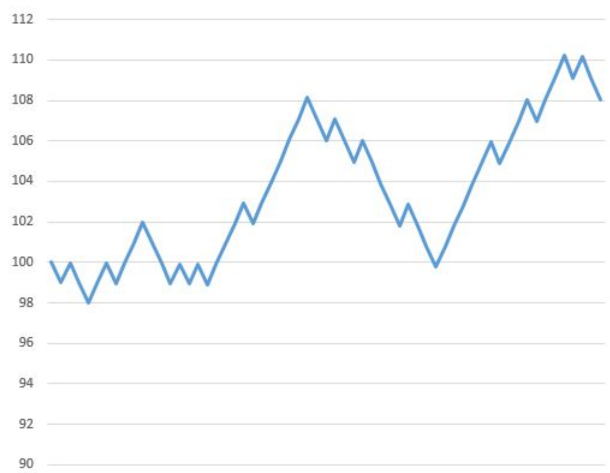

Figure 2.5 Binomial tree simulation path for 60 time steps

In contrast, we look at the movement of the European *EuroStoxx 50* index during three months in Figure 2.6.[7] At first sight the two figures look somehow similar but we can also clearly see differences: Most importantly it is visible that the binomial tree can only take certain values while the EuroStoxx has no such restrictions. Moreover, the EuroStoxx seems to vary more and looks less regular. We might conclude that the binomial tree is a good start and an easy-to-implement tool for simulating stocks. Still, there is scope for improvement. We will therefore present a different approach next.

[6]The fact that there is a difference can be mainly traced back to the finite sample (1.000) that is used for simulation, see e.g. [Gla03].
[7]Taken from google.com, retrieved January 23, 2018

Figure 2.6 EuroStoxx 50

Another disadvantage of the binomial tree model is that the set of time instants when the stock can move is limited. A more realistic model should thus at least allow for moving in continuous-time and extend the range of possible values. If we stick to the convention that the process of the risky asset $S(t)$ starts at $t = 0$, continuous time means that the domain is $\mathbb{R}^+$. What else could come to our mind when we think of a list of desirable properties?

(i) Having control over $S(0)$

The starting point $S(0)$ should be equal to the current value of the stock we want to simulate.[8] So $S(0)$ should be a degree of freedom in our model. If there were any restrictions on $S(0)$ our model would at most be feasible for some special situations but not in general.

(ii) The change of $S(t)$ over one time period is unrelated to the change of $S(t)$ in another time period.

Unrelated is not a mathematically defined notion. So we have to say that we understand by unrelatedness that the increments $S(t_3) - S(t_2)$ and $S(t_1) - S(t_0)$ are

[8] In mathematical terms we can only demand $S(0) = S_0$ almost surely. We do not discuss this measure theoretical peculiarity here and refer the reader to the literature, e.g. [Dur91].

stochastic independent for every quadruple $0 \leq t_0 \leq t_1 \leq t_2 \leq t_3 < \infty$.[9]

(iii) The increments have an appropriate behaviour.

This statement has also to be made more precise: At first, it does not make economically sense if the stock price can drop below 0 because nobody would be willing to pay for getting rid of a share.[10] A smooth way to circumvent negative prices is to consider the logarithm $W(t) = \log(S(t))$ and imposing conditions on it instead of $S(t)$, compare e.g. [CZ03], Section 3.2.2. As there are many different small influences on the price of a stock, central limit theorem then suggests that it makes sense to demand that the increments of $W(t)$ are normally distributed.

(iv) The paths of $S(t)$ are continuous.[11]

Finally, jumps in the process would very much complicate the mathematics behind the model and thus $S(t)$ should move continuously in time.

Indeed, it is possible to construct such processes $S(t)$ respectively $W(t)$, see e.g. [Øks03]. In these notes, we will not explain all details but only some of the most important aspects: The stochastic process $W(t)$ is a **Brownian motion** or **Wiener process**. The Wiener process $W(t)$ also satisfies the properties (i), (ii) and (iv). Moreover, at any point in time t the distribution of $W(t)$ is normal with expected value 0 and variance t. Hence, $W(t)$ can be simulated by drawing a number according to $\mathcal{N}(0, t)$. Furthermore the increments $W(t_1) - W(t_0)$ with $0 \leq t_0 \leq t_1$ are normally distributed. The expected value is again 0 and the variance is equal to $t_1 - t_0$.

The continuous-time stock price $S(t)$ can be shown to fulfill the **stochastic differential equation**

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t). \tag{2.1}$$

Although stochastic differential equations are rather complicated to define because they make use of the so-called **Itô-integral**, see e.g. [Øks03], we can give an easy interpretation of (2.1): The first summand corresponds to a **drift** or in other words to a global trend of the path. The higher μ is, the more pronounced this trend is. The parameter σ determines the **volatility** of the process. The higher σ is, the more the paths fluctuate. Moreover, it is even possible to give an explicit solution of this stochastic differential equation (compare [Øks03], Chapter 5). We will come back to a more detailed explanation of equation **??** after the following result.

---

[9]Recall that two random variables X, Y are stochastic independent if and only if the joint cumulative distribution function can be written as a product of the cumulative distribution functions of X and Y, i.e. $F_{X,Y} = F_X(x)F_Y(y)$.

[10]Only a few years ago, almost all economists would have agreed that negative interest rates do not makee sense because nobody would borrow money without compensation. Therefore, we have to be careful if this is really a realistic assumption.

[11]Also the paths are continuous only with probability 1 and not continuous in a deterministic sense.

Theorem 2.14

The solution of the stochastic differential equation (2.1) is

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right). \tag{2.2}$$

The solution $S(t)$ is a log-normally distributed random variable with expected value $S(0)e^{\mu t}$ and variance $S(0)^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$. The solution is called **geometric Brownian motion**.

Exercise 2.9

Let $S(0) = 100, \mu = 1, \sigma = 2$. Calculate the expected value and the variance of the corresponding geometric Brownian motion.

Exercise 2.10

Let $S(0) = 1$ and assume that the expected value and the variance are both equal to e at $t = 1$. Calculate μ and σ!

In order to implement geometric Brownian motion in excel, we need to re-formulate the stochastic differential equation (2.1) at first. Since excel can not simulate continuous time, we need to amend it to discrete time steps Δt, i.e. replace $\mathrm{d}S(t)$ by $S(t + \Delta t) - S(t)$ and so on such that we end up with

$$S(t + \Delta t) - S(t) = \mu S(t)\Delta t + S(t)\sigma(W(t + \Delta t) - W(t)).$$

Recall that the increment $W(t + \Delta t) - W(t)$ is normally distributed with mean 0 and variance Δt. If X is a standard random distributed variable, this leads to

$$S(t + \Delta t) - S(t) = \mu S(t)\Delta t + S(t)\sigma X\sqrt{\Delta t}.$$

This discrete formula can now be used in excel. This has been done in the excel template *Geometric_Browninan_Motion.xlsx*. Figure 2.7 shows a typical simulation path of the geometric Brownian motion. If we compare it to the EuroStoxx 50 chart in Figure 2.6 we see that the geometric Brownian motion very well achieves to imitate reality. This is the reason why it is commonly applied both in practice and theory to simulate the movement of a risky stock.

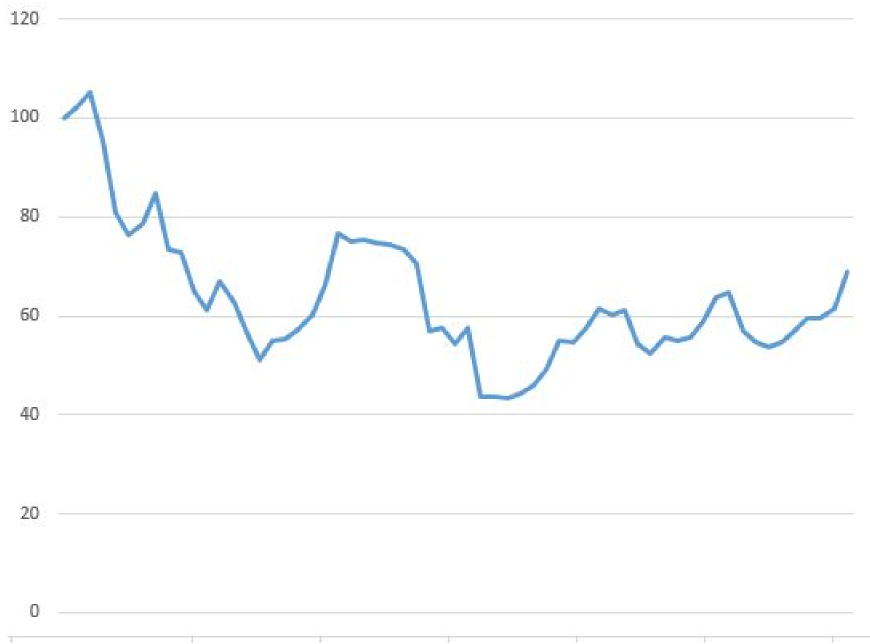

Figure 2.7 Geometric Brownian simulation path for 60 time steps

Still there are two main disadvantages of the geometric Brownian motion that have been discussed in the literature, see e.g. [Hul11]: The volatility in the model is assumed to be constant while we encounter times of high and times of low volatility in reality. Moreover, stock prices sometimes, though very rarely, do jump, for instance due to external shocks like changes in the interest rate or catastrophes.

3 Option pricing

We have just learned how to describe the movement of risky assets over time and found out that the geometric Brownian motion is a good instrument for that task. Still this does not tell us too much about how to find an appropriate *price* of a risky stock. In this chapter, we will see that the geometric Brownian motion is also a great tool for pricing. More precisely, we will apply the geometric Brownian motion in the context of option pricing. An option constitutes the right but no obligation to buy or sell an underlying asset at a specified price on a specified day.

We will at first give the basic definitions that are important regarding options and how to price an option within the simple binomial tree model. The technique that we will get to know is an application of the no-arbitrage principle. Finally, our presentation culminates in the famous **Black-Scholes formula** which is the main result of these notes. A main ingredient will be the geometric Brownian motion which we introduced in Section 2.3.

3.1 Basic definitions

> Whether you love derivatives or hate them, you cannot ignore them! The derivatives market is huge – much bigger than the stock market when measured in terms of underlying assets. The value of assets underlying derivatives is several times the world gross domestic products. ([Hul11], p.1)

This quote from [Hul11] wants to emphasize that it is worth to put some thought on derivatives. A *derivative* is a financial product which derives its value only from the performance of an underlying asset. Among the most known classes of derivatives are futures, swaps, forwards and options. Here, we concentrate on options. Since we do not assume that the reader is already familiar with the different types of options, we start with a brief overview over some of the most important aspects.

Before we go into more details, let us recall our market model: It consists of two assets, namely a (risk-free) bond $A(t)$ and a (risky) stock $S(t)$. A **call option** with

strike price K and **exercise time** t is a contract which gives the holder the right to *buy* a share of the stock for price K at time t.[1]

Example 3.1: Call option

Assume that the we are at $t = 0$*. The payoff profile of a call option* $C(t)$ *with exercise time is* 1 *and strike price* 100€ *depends on the (unknown) price* $S(1)$*. If* $S(1) \geq 100$*, then it is worth to exercise the option and the gain is* $S(1) - 100€$*. If* $S(1) < 100€$*, then it would not make sense to exercise the call and so the call yields* 0*. Hence its payoff function is*

$$C(1) = \begin{cases} S(1) - 100€ & \text{if } S(1) > 100€ \\ 0 & \text{else.} \end{cases}$$

Graphically, the payoff function corresponds to Figure 3.1.

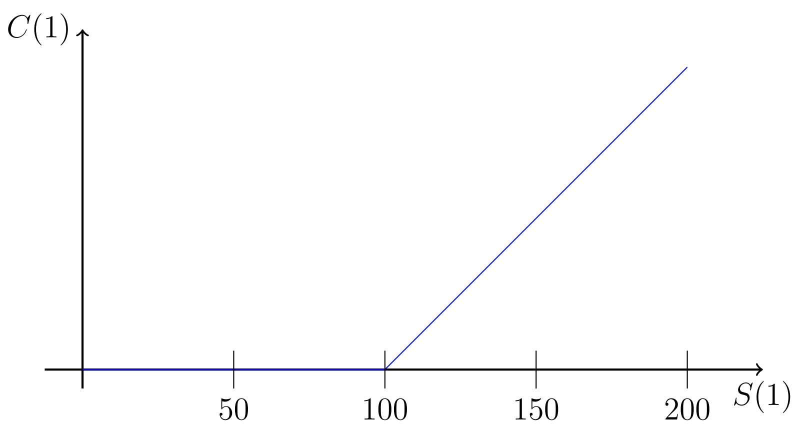

Figure 3.1 Payoff function of a call

In contrast, a **put option** with strike price $\widetilde{K}$ and exercise time t is a contract which gives the holder the right to *sell* a share of the stock for price $\widetilde{K}$ at time t.

Example 3.2: Put option

Also the payoff profile of a put option $P(t)$ *with exercise time* 1 *and strike price* 100€ *depends on* $S(1)$*. A put option is a bet that the stock will go down. If* $S(1) \leq 100€$*, then the option allows to sell the stock for* 100 *leading to a gain* $100€ - S(1)$*. If the stock price is high, i.e.* $S(1) > 100€$*, then the payoff of the*

[1] We draw our attention only to options which may be exercised at *one* point in time. These are called **European options**. Other famous types of options are **Bermuda** (several exercise dates) and **American** (exercise at any time before maturity).

put is 0€,

$$P(1) = \begin{cases} 0 & \text{if } S(1) > 100€ \\ 100€ - S(1) & \text{else.} \end{cases}$$

Figure 3.2 shows the payoff function.

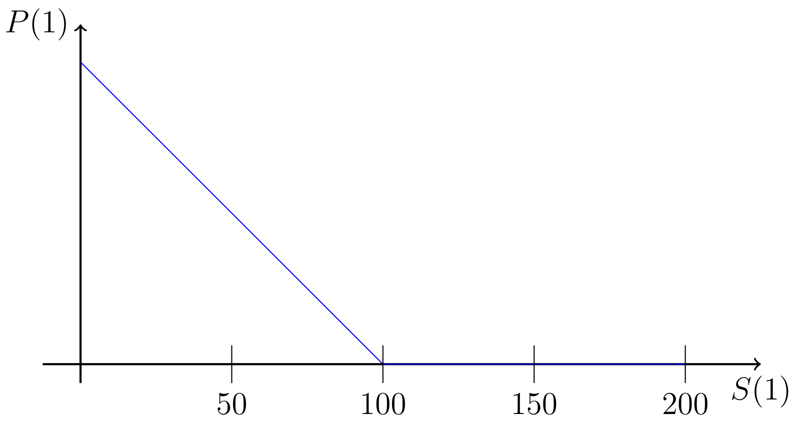

Figure 3.2 Payoff function of a put

Since options are traded on markets, they have a specific price people assign to them. If one thinks about it for a while, the following ideas may come to the mind:

- A put with a higher strike price is better than one with a lower strike price.
- A call with a higher strike price is worse than one with a lower strike price.
- If it is more likely that the stock will go up than down, buying a call seems to be a good investment

Although these rules are all helpful, they do not find an explicit price of any option. In order to remedy this shortcoming, we have to make assumptions about the movement of the stock. The first model we got to know was the (one-step) binomial tree model in Section 2.1. In this context, finding a price for an option is an application of the no-arbitrage principle from Section 1.2.

Example 3.3: Option pricing for binomial tree

Assume that $A(0) = S(0) = 100€$ *and that the risk-free interest rate is equal to* $r = 5\%$*. The stock* $S(1)$ *can either go up or down*

$$S(1) = \begin{cases} 110€ & \text{with probability } 50\% \\ 90€ & \text{with probability } 50\%, \end{cases}$$

as depcited in Figure 3.3.

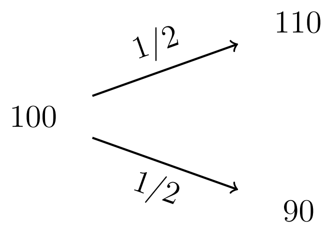

Figure 3.3 One-step binomial tree

We would now like to price a call option with exercise time 1 *and strike price* 100*. Also* $C(1)$ *is a random variable with*

$$C(1) = \begin{cases} 10€ & \text{if stock goes up} \\ 0€ & \text{if stock goes down.} \end{cases}$$

The question is now whether it is possible to replicate the payoff of $C(1)$ *by just using* $S(1)$ *and* $A(1)$*. In that case the law of one price, Theorem 1.6, applies. In mathematical terms we search for* $x, y \in \mathbb{R}$ *with*

$$xA(1) + yS(1) = C(1).$$

The payoff function of the left-hand side is

$$xS(1) + yA(1) = \begin{cases} 105x + 110y & \text{if stock goes up} \\ 105x + 90y & \text{if stock goes down.} \end{cases}$$

Being indifferent between the portfolio and the call no matter if the stock goes up or down, is equivalent to solving the linear equation system

$$\begin{aligned} 10 &= 105x + 110y \\ 0 &= 105x + 90y. \end{aligned}$$

Its solution is easy to calculate,

$$x = -\frac{3}{7}, \qquad y = \frac{1}{2}.$$

It is hence necessary to buy $\frac{1}{2}$ *shares of a risk-free bond and short sell* $\frac{3}{7}$ *shares of the stock. The payoff of the call is replicated at* $t = 1$ *in any case. By the law of one price, Theorem 1.6, the price of the option at* $t = 0$ *is thus*

$$C(0) = -\frac{3}{7}A(0) + \frac{1}{2}S(0) = \frac{100€}{14} \approx 7.14€.$$

Exercise 3.1

Assume that the prices $A(0), A(1), S(0)$ and $S(1)$ are as in Example 3.3. Calculate the price $P(0)$ of a put with exercise time 1 and strike 80€.

Exercise 3.2

Let $A(0) = S(0)$ and $S(1)$ be as in Example 3.3. Assume that the risk-free interest rate is equal to a constant $r > 0$. Give a general formula for the price of a call with exercise time $t = 1$ and strike 100€. Plot $P(1)(r)$ as a function of r.

As discussed in Section 2.3, the binomial tree is a good starting point for simulation but not at all completely satisfactory. Even the geometric Brownian motion has some drawbacks and the same is true for other asset models like the *Hull-White model*, the *Vasicek model* or the *Black-Karasinski model*. It would therefore be nice to find a general rule not depending on the asset model which explains the prices of options. This can be done at least partially.

Example 3.4: Put-Call-Parity

The market interest is $r = 10\%$ and $A(0) = 100€$. A trader buys a call and short sells a put at $t = 0$. Both options have exercise date $T = 1$ and strike price 110€. The aggregated payoff is

$$\begin{aligned} C(1) - P(1) &= \max(S(1) - 110€, 0) - \max(110€ - S(1), 0) \\ &= S(1) - 110€ = S(1) - (1 + r)100€ = S(1) - A(1). \end{aligned}$$

This means, that no matter how the stock will evolve, buying a call and short selling a put has the same payoff as buying a stock and short selling a bond. It is very remarkable that the equation does not depend on the model of the stock price. The law of one price, Tbeorem 1.6 implies that also the prices at $t = 0$ coincide, i.e.

$$C(0) - P(0) = S(0) - A(0). \tag{3.1}$$

Exercise 3.3

Reconsider Example 3.4 and assume in addition that $S(0) = 80€$. What would be the market price of a call if the price of a put is $40,50€$?

Before it is possible to state the so-called put-call-parity in its general form, there is one last ingredient missing. Recall that passing from discrete time interest (paid once or severtal times during a year) to continuous time interest (paid continuously with a constant rate) corresponds to multiplying $A(0)$ by e^{rt} instead of multiplying by just $(1+r)^t$. For the mathematical reasons we refer to e.g. [HW18], Section 3.5.

Theorem 3.5: Put-Call-Parity

Let $C(t)$ be a call and $P(t)$ be a put both with exercise time T and strike price K. Then the following equation holds

$$C(0) + Ke^{-rT} = P(0) + S(0).$$

Note that the calculation we did in Example 3.4 also works as a general proof of Theorem 3.5.[2]

Exercise 3.4

(Compare [Rob17]) You are convinced that the stock of the *Latvian-German company* is going to have a huge movement within the next three months. However, you are not sure whether it will go up or down. The current price of the stock is $S(0) = 100$€. A three-month call option at an exercise price 100€ can be purchased for 10€.

a) Suppose that the risk-free interest rate is $r = 10\%$. Calculate the price of a three-month put option on the *Latvian-German company* stock at an exercise price of 100€.

b) Find a simple options strategy that resembles your expectations about the future movement of the stock price. How far would the stock have to move in either direction to make your strategy succeed?

3.2 The Black-Scholes Formula

The put-call-parity, Theorem 3.5, is an application of the no-arbitrage principle and implies that there is a direct connection between the price of a put and the price of a call. Therefore, it suffices to analyze either of them. In the following, we restrict our attention to calls. Furthermore let us point out, that we will not treat the topic in a mathematically fully precise way since this would make it necessary to study stochastic calculus and in particular the Itô-integral. For the mathematical details we refer the reader to [Kal17]. Nevertheless, it is possible to state the Black-Scholes formula in its correct form even without this deep foundation.

At first, we need to introduce the **Black-Scholes model**: Assume that the stock

[2]We swept under the rug a hidden condition of the put-call-parity, namely that the stock is not going to pay any dividends.

price evolves according to a geometric Brownian motion. In Section 2.3 respectively (2.2), we discussed that the stock price at time t is given by

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right),$$

where $W(t)$ is a standard Wiener process. Recall that $W(t_0)$ is normally distributed with expected value 0 and variance t_0. In order to keep the presentation clear, it turns out to be useful to use the abbreviation $m := \mu - \frac{\sigma^2}{2}$, such that we have

$$S(t) = S(0) \exp\left(mt + \sigma W(t)\right).$$

As we found out in Section 2.2, a rational investor should price the call according to the risk-neutral measure. How do we find this measure? A necessary condition is that the risk-neutral expectation E_* of the discounted stock price $e^{-rt}S(t)$ is constant.[3] The Black-Scholes formula prices the call option C under the risk-neutral measure P_*.

Theorem 3.6: Black-Scholes formula

Let C be a call option with strike price K and exercise date T. Then its fair price is

$$C(0) = S(0)\Phi_{0,1}(d_1) - Ke^{-rT}\Phi_{0,1}(d_2),$$

where $\Phi_{0,1}$ is the cumulative distribution function of the normal distribution and

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

The Black-Scholes formula is widely used in practice. Indeed, the formula led to a boom in options trading in the 1970s after its publication. And it is almost a self-fulfilling prophecy: the option prices we nowadays observe on the market are very close to what the Black-Scholes formula suggests (they were not in the past). According to [Ste13], the Black-Scholes formula is for that reason even one of the seventeen formulas that changed the course of the world.

Example 3.7

Assume that the risk-free market interest rate is $r = 3\%$ and that the volatility is $\sigma = 0.4$. The current price of the stock is 100€. What is the price of a call with

[3] This condition is a consequence of the so-called **martingale property**.

exercise date 2 *years and strike price* 80€*? At first, we need to find*

$$d_1 = \frac{\ln\left(\frac{100}{80}\right) + \left(0.03 + \frac{1}{2}0.4^2\right)2}{0.4\sqrt{2}} \approx 0.7834$$
$$d_2 = \frac{\ln\left(\frac{100}{80}\right) + \left(0.03 - \frac{1}{2}0.4^2\right)2}{0.4\sqrt{2}} \approx 0.2177.$$

Next we apply the cumulative distribution function of the standard normal distribution

$$\Phi_{0,1}(d_1) \approx 0.7833$$
$$\Phi_{0,1}(d_2) \approx 0.5862.$$

Now everything is at hand to calculate the Black-Scholes price of the call

$$C(0) \approx 100 \cdot 0.7833 - 80 \cdot e^{-0.03 \cdot 2} 0.5862 \approx 34.17$$

The market price of the option is thus approximately 34.17€*.*

Exercise 3.5

Assume that the risk-free market interest rate is $r = 4\%$ and that the volatility is $\sigma = 0.4$. The current price of the stock is 60€. What is the price of a call with exercise time 2 years and strike price 80€? Compare it to the price in Example 3.7 and explain why your result makes sense.

Exercise 3.6

Let the market conditions be as in Example 3.7. Calculate the Black-Scholes price of a put with exercise time 5 years and strike price 90€.

In the excel template *Black_Scholes_Formula.xlsx*, the Black-Scholes formula has been implemented. As input the necessary market data is used and as output excel calculates d_1, d_2 and the Black-Scholes price $C(0)$ of the call option.

Exercise 3.7

Amend *Black_Scholes_Formula.xlsx* such that it can also calculate the price of a put.

At the very end of these notes, let us also discuss some drawbacks of the Black-Scholes model: First, it is based on the geometric Brownian motion and so all its shortcomings are transferred to the Black-Scholes model. Most importantly, the market volatility is assumed to be constant. In reality however, the typical shape of market volatility depends on both strike price and exercise time, i.e. the graph of implied volatility against against strike and maturity is not flat. The real-life market volatility rather looks like a smiling mouth, the so-called **volatility smile**, compare Figure 3.4.

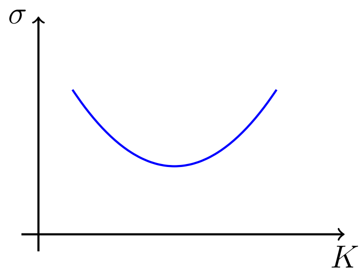

Figure 3.4 Volatility Smile

Besides the volatility, also the interest rate is assumed to be constant. This is for sure not the case in reality but interest rate moves over time. Especially long-term options are affected by this, or as the investor Warren Buffet wrote in [Ber09]:

> I believe the Black-Scholes formula, even though it is the standard for establishing the dollar liability for options, produces strange results when the long-term variety are being valued... The Black-Scholes formula has approached the status of holy writ in finance ... If the formula is applied to extended time periods, however, it can produce absurd results. In fairness, Black and Scholes almost certainly understood this point well. But their devoted followers may be ignoring whatever caveats the two men attached when they first unveiled the formula.

Exercise 3.8

This exercise might be a bit tough: Calculate the expectation

$$E\left(e^{-rt}S(t)\right)f = S(0)E\left(e^{\sigma W(t)+(m-r)t}\right)$$

with real market probabilities. It might be useful to recap Example 2.10 before starting with the calculation.

References

[Alb07] P. Albrecht, *Grundprinzipien der Finanz- und Versicherungsmathematik: Grundlagen und Anwendungen der Bewertung von Zahlungsströmen*, Schäffer-Poeschel, 2007.

[Bar17] N. Barraco, *Parametric Error in LSMC Methodology*, Master Thesis, Universität Ulm, 2017.

[Ber09] Berkshire Hathaway, *Berkshire's Corporate Performance vs. S&P 500*, 2009.

[Boe18] Börsen-Zeitung, *Länder-Ratings*, Retrieved January 16, 2018 from `https://www.boersen-zeitung.de/index.php?li=312&subm=laender`.

[Bus18] Business Dictionary, *Risk*, Retrieved January 19, 2018 from `http://www.businessdictionary.com/definition/risk.html`.

[CZ03] M. Capiński, T. Zastawniak,, *Mathematics for Finance*, Springer, 2003.

[Dur91] R. Durrett, *Probability: Theory and Examples*, Wadsworth & Brooks/Cole, 1991.

[Eco18] Economic Times, *Definition of 'Risk'*, Retrieved January 19, 2018 from `https://economictimes.indiatimes.com/definition/risk`.

[Eid17] Eidgenössische Finanzaufsicht FINMA, "Schweizer Solvenztest (SST)", Rundschreiben 2017/3, 2017.

[EC13] European Commission, *Commission Delegated Regulation (EU) 2013/575*, 2013.

[EC15] European Commission, *Commission Delegated Regulation (EU) 2015/35*, 2015.

[Fin06] C. Fine, *A mind on its own: How your Brain Distorts and Deceives*, Icon Books, 2006.

[Gla03] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer, 2003.

[HW18] A. Hirn, C. Weiß, *Analysis - Grundlagen und Exkurse: Grundprinzipien der Differential- und Integralrechnung*, Springer, 2018.

[Hul11] J. Hull, *Options, Futures and other Derivatives*, Pearson, 2011.

[Kah12] D. Kahneman, *Thinking, Fast and Slow*, Penguin, 2012.

[Kal17] J. Kallsen, *Mathematical finance: An introduction in discrete time*, Lecture Notes, 2017.

[Mar52] H. Markowitz, *Portfolio Selection*, In: Journal of Finance, 72, p. 77–92

[MFE05] A. McNeil, R. Frey, P. Embrechts, *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton Series in Finance, 2005.

[Øks03] B. Øksendal, B., *Stochastic Differential Equations*, Springer, 2003.

[Rob17] M. Robe, *Practice Set #5 and Solutions*, Retrieved January 24, 2018 from `http://www1.american.edu/academic.depts/ksb/finance_realestate/mrobe/465/PS/PS_5_04.pdf`.

[Sen77] A. Sen, *Rational Fools: A Critique of the Behavioural Foundations of Economic Theory*, in: Philosophy and Public Affairs, 317, 1977.

[Ste13] I. Stewart, *Seventeen Equations that Changed the World*, Profile Books, 2013.